A TOPOLOGY ON E-THEORY

JOINT WORK WITH C. SCHAFHAUSER

José Carrión
TCU

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The E-theory group $E(A, B)$

- bifunctor from sep. C^*-algebras to abelian groups, defined by Connes–Higson (1990)
- stable, homotopy invariant
- has composition product $E(A, B) \times E(B, C) \to E(A, C)$

Parallel to $KK(A, B)$, but:

- is half-exact; has 6-term exact sequences in each variable (only true for KK under nuclearity assumption)

Asymptotic morphisms $\varphi\colon A \xrightarrow{\approx} B$

- family $(\varphi_t)_{t\in[0,\infty)}$ of functions $A \to B$
- $t \mapsto \varphi_t(a)$ is continuous $\forall a \in A$
- asymptotically satisfies properties of *-homomorphisms ($\|\varphi_t(a+\lambda b) - \varphi_t(a) - \lambda\varphi_t(b)\| \to 0$, etc.)

Can define notion of homotopy for asymptotic morphisms.

Definition

- $[[A, B]] := \{A \xrightarrow{\approx} B\}/\text{homotopy}$
- $E(A, B) := [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$

Fact: $KK(A, B) \cong E(A, B)$ if A is nuclear.

THE TOPOLOGIES ON $KK(A, B)$, $E(A, B)$

Some history

- the topology on $KK(A, B)$ can be traced back to Brown–Douglas–Fillmore ('77)
- was first studied in depth by L. Brown ('84), then Salinas ('92), in connection with quasidiagonality
- further developed by Schochet ('01), before being defined in general by Pimsner (unpublished) and Dadarlat ('05)

The "Hausdorffization" of $KK(A, B)$: $KL(A, B)$

- important in classification theory of nuclear C^*-algebras
- first defined by Rørdam, under UCT assumption
- Dadarlat defined in general as $KK(A, B)/\overline{\{0\}}$

Theorem (C–Schafhauser)

There is a unique second-countable topology on $E(A,B)$ s.t.

$$x_n \to x \in E(A,B) \quad \Leftrightarrow \quad \exists\, y \in E\big(A, C(\mathbb{N}^\dagger, B)\big) \text{ s.t.}$$
$$y(n) = x_n \text{ and } y(\infty) = x \;\; \forall n \in \mathbb{N}$$

(known as "Pimsner's condition").

Moreover, the product $E(A,B) \times E(B,C) \to E(A,C)$ is continous (just like for KK).

Define $EL(A, B) := E(A, B)/\overline{\{0\}}$.

Theorem (C–Schafhauser)

$EL(A, B)$ is a separable, totally disconnected, completely metrizable topological group.

Theorem (C–Schafhauser)

$$EL(\varinjlim A_n, B) \cong \varprojlim EL(A_n, B)$$

so

$$KL(\varinjlim A_n, B) \cong \varprojlim KL(A_n, B)$$

when all A_n are nuclear.

METHODS USED

- Recall: $[[A, B]] \coloneqq \{A \xrightarrow{\approx} B\}/\text{homotopy}$.
- We study $[[A, B]]$ and define topology on it. Topology on $E(A, B)$ is special case.
- Define and study $[[A, B]]_{\mathrm{Hd}}$, the "Hausdorffization" of $[[A, B]]$. $EL(A, B)$ is special case.
- Main tool: Blackadar's shape theory and its connection with asymptotic morphisms, discovered by Dadarlat ('94).

SHAPE THEORY, SEMIPROJECTIVITY

- General idea: write a C^*-algebra A as a direct limit

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \to A$$

  of "nice" C^*-algebras, studying properties of A determined (up to homotopy equivalence) by the inductive system.
- "Nice": *semiprojective*, an analog of ANR for topological spaces.
- Not known if every C^*-algebra can be written as a limit of semiprojective C^*-algebras.
- *However:* every C^*-algebra can be written as such a limit with semiprojective connecting maps (a *shape system*).

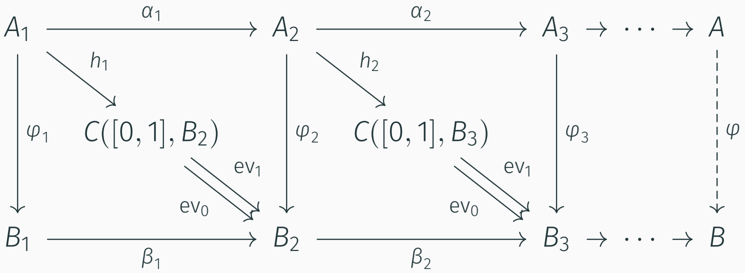
The shape category SH of Blackadar–Dadarlat

Objects: shape systems (A_n, α_n)

Morphisms:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & \circlearrowleft_h & \downarrow & \circlearrowleft_h & \downarrow & & \\
\cdots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \cdots
\end{array}$$

Getting $\varphi : A \xrightarrow{\approx} B$ from a morphism in SH:



Theorem (Dadarlat)

Every asymptotic morphism $\varphi : A \xrightarrow{\approx} B$ arises from a morphism of shape systems.

Theorem (C–Schafhauser)

There is an equivalence of categories $\mathrm{SH} \xrightarrow{\sim} \mathrm{AM}_{\mathrm{Hd}}$, where $\mathrm{AM}_{\mathrm{Hd}}$ is the category whose objects are separable C^*-algebras and where the morphisms between A and B are the elements of $[[A, B]]_{\mathrm{Hd}}$.

THANK YOU!